BASIC MATHEMATICS

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To Jerry

My publishers, Addison-Wesley, have produced my books for these last eight years. I want it known how much I appreciate their extraordinary performance at all levels. General editorial advice, specific editing of the manuscripts, and essentially flawless typesetting and proof sheets. It is very gratifying to have found such a company to deal with.

New York, 1970

Serge Lang

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S.L.

Foreword

The present book is intended as a text in basic mathematics. As such, it can have multiple use: for a one-year course in the high schools during the third or fourth year (if possible the third, so that calculus can be taken during the fourth year); for a complementary reference in earlier high school grades (elementary algebra and geometry are covered); for a one-semester course at the college level, to review or to get a firm foundation in the basic mathematics necessary to go ahead in calculus, linear algebra, or other topics.

Years ago, the colleges used to give courses in "college algebra" and other subjects which should have been covered in high school. More recently, such courses have been thought unnecessary, but some experiences I have had show that they are just as necessary as ever. What is happening is that the colleges are getting a wide variety of students from high schools, ranging from exceedingly well-prepared ones who have had a good first course in calculus, down to very poorly prepared ones. This latter group includes both adults who return to college after several years' absence in order to improve their technical education, and students from the high schools who were not adequately taught. This is the reason why some material properly belonging to the high-school level must still be offered in the colleges.

The topics in this book are covered in such a way as to bring out clearly all the important points which are used afterwards in higher mathematics. I think it is important not to separate arbitrarily in different courses the various topics which involve both algebra and geometry. Analytic geometry and vector geometry should be considered simultaneously with algebra and plane geometry, as natural continuations of these. I think it is much more valuable to go into these topics, especially vector geometry, rather than to go endlessly into more and more refined results concerning triangles or trigonometry, involving more and more complicated technique. A minimum of basic techniques must of course be acquired, but it is better to extend these techniques by applying them to new situations in which they become motivated, especially when the possible topics are as attractive as vector geometry.

In fact, for many years college courses in physics and engineering have faced serious drawbacks in scheduling because they need simultaneously some calculus and also some vector geometry. It is very unfortunate that the most basic operations on vectors are introduced at present only in college. They should appear at least as early as the second year of high school. I cannot write here a text for elementary geometry (although to some extent the parts on intuitive geometry almost constitute such a text), but I hope that the present book will provide considerable impetus to lower considerably the level at which vectors are introduced. Within some foreseeable future, the topics covered in this book should in fact be the standard topics for the second year of high school, so that the third and fourth years can be devoted to calculus and linear algebra.

If only preparatory material for calculus is needed, many portions of this book can be omitted, and attention should be directed to the rules of arithmetic, linear equations (Chapter 2), quadratic equations (Chapter 4), coordinates (the first three sections of Chapter 8), trigonometry (Chapter 11), some analytic geometry (Chapter 12), a simple discussion of functions (Chapter 13), and induction (Chapter 16, §1). The other parts of the book can be omitted. Of course, the more preparation a student has, the more easily he will go through more advanced topics.

"More preparation", however, does not mean an accumulation of technical material in which the basic ideas of a subject are completely drowned. I am always disturbed at seeing endless chains of theorems, most of them of no interest, and without any stress on the main points. As a result, students do not remember the essential features of the subject. I am fully aware that because of the pruning I have done, many will accuse me of not going "deeply enough" into some subjects. I am quite ready to confront them on that. Besides, as I prune some technical and inessential parts of one topic, I am able to include the essential parts of another topic which would not otherwise be covered. For instance, what better practice is there with negative numbers than to introduce at once coordinates in the plane as a pair of numbers, and then deal with the addition and subtraction of such pairs, componentwise? This introduction could be made as early as the fourth grade, using maps as a motivation. One could do roughly what I have done here in Chapter 8, §1, Chapter 9, §1, and the beginning of Chapter 9, §2 (addition of pairs of numbers, and the geometric interpretation in terms of a parallelogram). At such a level, one can then leave it at that.

The same remark applies to the study of this book. The above-mentioned sections can be covered very early, at the same time that you study numbers and operations with numbers. They give a very nice geometric flavor to a slightly dry algebraic theory.

Generally speaking, I hope to induce teachers to leave well enough alone, and to avoid torturing a topic to death. It is easier to advance in one topic by going ahead with the more elementary parts of another topic, where the first one is applied. The brain much prefers to work that way, rather than to concentrate on ugly technical formulas which are obviously unrelated to anything except artificial drilling. Of course, some rote drilling is necessary. The problem is how to strike a balance. Do not regard some lists of exercises as too short. Rather, realize that practice for some notion may come again later in conjunction with another notion. Thus practice with square roots comes not only in the section where they are defined, but also later when the notion of distance between points is discussed, and then in a context where it is more interesting to deal with them. The same principle applies throughout the book.

The Interlude on logic and mathematical expression can be read also as an introduction to the book. Because of various examples I put there, and because we are already going through a Foreword, I have chosen to place it physically somewhat later. Take a look at it now, and go back to it whenever you feel the need for such general discussions. Mainly, I would like to make you feel more relaxed in your contact with mathematics than is usually the case. I want to stimulate thought, and do away with the general uptight feelings which people often have about math. If, for instance, you feel that any chapter gets too involved for you, then skip that part until you feel the need for it, and look at another part of the book. In many cases, you don't necessarily need an earlier part to understand a later one. In most cases, the important thing is to have understood the basic concepts and definitions, to be at ease with the simpler computational aspects of these concepts, and then to go ahead with a more advanced topic.

This advice also applies to the book as a whole. If you find that there is not enough material in this book to occupy you for a whole year, then start studying calculus or possibly linear algebra.

The book deals with mathematics on both the manipulative (or computational) level and the theoretical level. You must realize that a mastery of mathematics involves both levels, although your tastes may direct you more strongly to one or the other, or both. Here again, you may wish to vary the emphasis which you place on them, according to your needs or your taste. Be warned that deficiency at either level can ultimately hinder you in your work. Independently of need, however, it should be a source of pleasure to understand why a mathematical result is true, i.e. to understand its proof as well as to understand how to use the result in concrete circumstances.

Try to rely on yourself, and try to develop a trust in your own judgment. There is no "right" way to do things. Tastes differ, and this book is not meant to suppress yours. It is meant to propose some basic mathematical topics, according to my taste. If I am successful, you will agree with my taste, or you will have developed your own.

New York S.L.
January 1971

Contents

Part One
ALGEBRA

In this part we develop systematically the rules for operations with numbers, relations among numbers, and properties of these operations and relations: addition, multiplication, inequalities, positivity, square roots, n-th roots. We find many of them, like commutativity and associativity, which recur frequently in mathematics and apply to other objects. They apply to complex numbers, but also to functions or mappings (in this case, commutativity does not hold in general and it is always an interesting problem to determine when it does hold).

Even when we study geometry afterwards, the rules of algebra are still used, say to compute areas, lengths, etc., which associate numbers with geometric objects. Thus does algebra mix with geometry.

The main point of this chapter is to condition you to have efficient reflexes in handling addition, multiplication, and division of numbers. There are many rules for these operations, and the extent to which we choose to assume some, and prove others from the assumed ones, is determined by several factors. We wish to assume those rules which are most basic, and assume enough of them so that the proofs of the others are simple. It also turns out that those which we do assume occur in many contexts in mathematics, so that whenever we meet a situation where they arise, then we already have the training to apply them and use them. Both historical experience and personal experience have gone into the selection of these rules and the order of the list in which they are given. To some extent, you must trust that it is valuable to have fast reflexes when dealing with associativity, commutativity, distributivity, cross-multiplication, and the like, if you do not have the intuition yourself which makes such trust unnecessary. Furthermore, the long list of the rules governing the above operations should be taken in the spirit of a description of how numbers behave.

It may be that you are already reasonably familiar with the operations between numbers. In that case, omit the first chapter entirely, and go right ahead to Chapter 2, or start with the geometry or with the study of coordinates in Chapter 7. The whole first part on algebra is much more dry than the rest of the book, and it is good to motivate this algebra through geometry. On the other hand, your brain should also have quick reflexes when faced with a simple problem involving two linear equations or a quadratic equation. Hence it is a good idea to have isolated these topics in special sections in the book for easy reference.

In organizing the properties of numbers, I have found it best to look successively at the integers, rational numbers, and real numbers, at the cost of slight repetitions. There are several reasons for this. First, it is a good way of learning certain rules and their consequences in a special context (e.g. associativity and commutativity in the context of integers), and then observing that they hold in more general contexts. This sort of thing happens very frequently in mathematics. Second, the rational numbers provide a wide class of numbers which are used in computations, and the manipulation of fractions thus deserves special emphasis. Third, to follow the sequence integers–rational numbers–real numbers already plants in your mind a pattern which you will encounter again in mathematics. This pattern is related to the extension of one system of objects to a larger system, in which more equations can be solved than in the smaller system. For instance, the equation $2x = 3$ can be solved in the rational numbers, but not in the integers. The equations $x^2 = 2$ or $10^x = 2$ can be solved in the real numbers but not in the rational numbers. Similarly, the equations $x^2 = -1$, or $x^2 = -2$, or $10^x = -3$ can be solved in the complex numbers but not in the real numbers. It will be useful to you to have met the idea of extending mathematical systems at this very basic stage because it exhibits features in common with those in more advanced contexts.

1 Numbers

§1. THE INTEGERS

The most common numbers are those used for counting, namely the numbers

$$1, 2, 3, 4, \ldots,$$

which are called the **positive integers.** Even for counting, we need at least one other number, namely,

$$0 \text{ (**zero**)}.$$

For instance, we may wish to count the number of right answers you may get on a test for this course, out of a possible 100. If you get 100, then all your answers were correct. If you get 0, then no answer was correct.

The positive integers and zero can be represented geometrically on a line, in a manner similar to a ruler or a measuring stick:

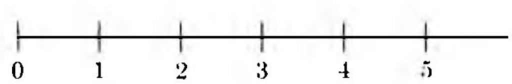

Fig. 1–1

For this we first have to select a unit of distance, say the inch, and then on the line we mark off the inches to the right as in the picture.

For convenience, it is useful to have a name for the positive integers together with zero, and we shall call these the **natural numbers.** Thus 0 is a natural number, so is 2, and so is 124,521. The natural numbers can be used to measure distances, as with the ruler.

By definition, the point represented by 0 is called the **origin.**

The natural numbers can also be used to measure other things. For example, a thermometer is like a ruler which measures temperature. However, the thermometer shows us that we encounter other types of numbers besides the natural numbers, because there may be temperatures which may go below 0. Thus we encounter naturally what we shall call **negative integers** which we call **minus** 1, **minus** 2, **minus** 3, . . . , and which we write as

$$-1, \ -2, \ -3, \ -4, \ldots .$$

We represent the negative integers on a line as being on the other side of 0 from the positive integers, like this:

Fig. 1-2

The positive integers, negative integers, and zero all together are called the **integers.** Thus -9, 0, 10, -5 are all integers.

If we view the line as a thermometer, on which a unit of temperature has been selected, say the degree Fahrenheit, then each integer represents a certain temperature. The negative integers represent temperatures below zero.

Our discussion is already typical of many discussions which will occur in this course, concerning mathematical objects and their applicability to physical situations. In the present instance, we have the integers as mathematical objects, which are essentially abstract quantities. We also have different applications for them, for instance measuring distance or temperatures. These are of course not the only applications. Namely, we can use the integers to measure time. We take the origin 0 to represent the year of the birth of Christ. Then the positive integers represent years after the birth of Christ (called AD years), while the negative integers can be used to represent BC years. With this convention, we can say that the year -500 is the year 500 BC.

Adding a positive number, say 7, to another number, means that we must move 7 units to the right of the other number. For instance,

$$5 + 7 = 12.$$

Seven units to the right of 5 yields 12. On the thermometer, we would of course be moving upward instead of right. For instance, if the temperature at a given time is 5° and if it goes up by 7°, then the new temperature is 12°.

Observe the very simple rule for addition with 0, namely

N1. $$\boxed{0 + a = a + 0 = a}$$

for any integer a.

What about adding negative numbers? Look at the thermometer again. Suppose the temperature at a given time is 10°, and the temperature drops by 15°. The new temperature is then $-5°$, and we can write

$$10 - 15 = -5.$$

Thus -5 is the result of subtracting 15 from 10, or of adding -15 to 10.

In terms of points on a line, adding a negative number, say -3, to another number means that we must move 3 units to the left of this other number. For example,

$$5 + (-3) = 2$$

because starting with 5 and moving 3 units to the left yields 2. Similarly,

$$7 + (-3) = 4, \qquad \text{and} \qquad 3 + (-5) = -2.$$

Note that we have

$$3 + (-3) = 0 \qquad \text{or} \qquad 5 + (-5) = 0.$$

We can also write these equations in the form

$$(-3) + 3 = 0 \qquad \text{or} \qquad (-5) + 5 = 0.$$

For instance, if we start 3 units to the left of 0 and move 3 units to the right, we get 0. Thus, in general, we have the formulas (by assumption):

N2. $$\boxed{a + (-a) = 0 \qquad \text{and also} \qquad -a + a = 0.}$$

In the representation of integers on the line, this means that a and $-a$ lie on opposite sides of 0 on that line, as shown on the next picture:

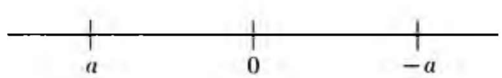

Fig. 1–3

Thus according to this representation we can now write

$$3 = -(-3) \qquad \text{or} \qquad 5 = -(-5).$$

In these special cases, the pictures are:

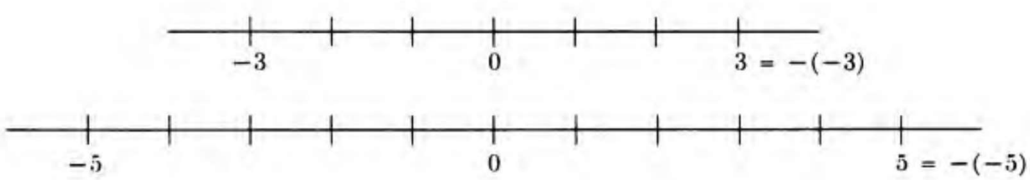

Fig. 1–4

Remark. We use the name

$$\textbf{minus } \boldsymbol{a} \qquad \text{for} \qquad \boldsymbol{-a}$$

rather than the words "negative a" which have found some currency recently. I find the words "negative a" confusing, because they suggest that $-a$ is a negative number. This is not true unless a itself is positive. For instance,

$$3 = -(-3)$$

is a positive number, but 3 is equal to $-a$, where $a = -3$, and a is a negative number.

Because of the property

$$a + (-a) = 0,$$

one also calls $-a$ the **additive inverse** of a.

The sum and product of integers are also integers, and the next sections are devoted to a description of the rules governing addition and multiplication.

§2. RULES FOR ADDITION

Integers follow very simple rules for addition. These are:

Commutativity. *If a, b are integers, then*

$$\boxed{a + b = b + a.}$$

For instance, we have

$$3 + 5 = 5 + 3 = 8,$$

or in an example with negative numbers, we have

$$-2 + 5 = 3 = 5 + (-2).$$

Associativity. *If a, b, c are integers, then*

$$\boxed{(a + b) + c = a + (b + c).}$$

In view of this, it is unnecessary to use parentheses in such a simple context, and we write simply

$$a + b + c.$$

For instance,

$$(3 + 5) + 9 = 8 + 9 = 17,$$
$$3 + (5 + 9) = 3 + 14 = 17.$$

We write simply

$$3 + 5 + 9 = 17.$$

Associativity also holds with negative numbers. For example,

$$(-2 + 5) + 4 = 3 + 4 = 7,$$
$$-2 + (5 + 4) = -2 + 9 = 7.$$

Also,

$$(2 + (-5)) + (-3) = -3 + (-3) = -6,$$
$$2 + (-5 + (-3)) = 2 + (-8) = -6.$$

The rules of addition mentioned above will not be proved, but we shall prove other rules from them.

To begin with, note that:

N3. $\boxed{\textit{If } a + b = 0, \textit{ then } b = -a \textit{ and } a = -b.}$

To prove this, add $-a$ to both sides of the equation $a + b = 0$. We get

$$-a + a + b = -a + 0 = -a.$$

Since $-a + a + b = 0 + b = b$, we find

$$b = -a$$

as desired. Similarly, we find $a = -b$. We could also conclude that

$$-b = -(-a) = a.$$

As a matter of convention, we shall write

$$a - b$$

instead of

$$a + (-b).$$

Thus a sum involving three terms may be written in many ways, as follows:

$$\begin{aligned}(a-b)+c &= (a+(-b))+c \\ &= a+(-b+c) && \text{by associativity} \\ &= a+(c-b) && \text{by commutativity} \\ &= (a+c)-b && \text{by associativity,}\end{aligned}$$

and we can also write this sum as

$$a-b+c = a+c-b,$$

omitting the parentheses. Generally, in taking the sum of integers, we can take the sum in any order by applying associativity and commutativity repeatedly.

As a special case of **N3,** for any integer a we have

N4. $$\boxed{a = -(-a).}$$

This is true because

$$a + (-a) = 0,$$

and we can apply **N3** with $b = -a$. Remark that this formula is true whether a is positive, negative, or 0. If a is positive, then $-a$ is negative. If a is negative, then $-a$ is positive. In the geometric representation of numbers on the line, a and $-a$ occur symmetrically on the line on opposite sides of 0. Of course, we can pile up minus signs and get other relationships, like

$$-3 = -(-(-3)),$$

or

$$3 = -(-3) = -(-(-(-3))).$$

Thus when we pile up the minus signs in front of a, we obtain a or $-a$ alternatively. For the general formula with the appropriate notation, cf. Exercises 5 and 6 of §4.

From our rules of operation we can now prove:

For any integers a, b we have

$$-(a+b) = -a + (-b)$$

or, in other words,

N5.
$$\boxed{-(a + b) = -a - b.}$$

Proof. Remember that if x, y are integers, then $x = -y$ and $y = -x$ mean that $x + y = 0$. Thus to prove our assertion, we must show that

$$(a + b) + (-a - b) = 0.$$

But this comes out immediately, namely,

$$\begin{aligned}(a + b) + (-a - b) &= a + b - a - b &&\text{by associativity}\\ &= a - a + b - b &&\text{by commutativity}\\ &= 0 + 0\\ &= 0.\end{aligned}$$

This proves our formula.

Example. We have

$$\begin{aligned}-(3 + 5) &= -3 - 5 = -8,\\ -(-4 + 5) &= -(-4) - 5 = 4 - 5 = -1,\\ -(3 - 7) &= -3 - (-7) = -3 + 7 = 4.\end{aligned}$$

You should be very careful when you take the negative of a sum which involves itself in negative numbers, taking into account that

$$-(-a) = a.$$

The following rule concerning positive integers is so natural that you probably would not even think it worth while to take special notice of it. We still state it explicitly.

If a, b are positive integers, then $a + b$ is also a positive integer.

For instance, 17 and 45 are positive integers, and their sum, 62, is also a positive integer.

We assume this rule concerning positivity. We shall see later that it also applies to positive real numbers. From it we can prove:

If a, b are negative integers, then $a + b$ is negative.

Proof. We can write $a = -n$ and $b = -m$, where m, n are positive. Therefore

$$a + b = -n - m = -(n + m),$$

which shows that $a + b$ is negative, because $n + m$ is positive.

Example. If we have the relationship between three numbers

$$\boxed{a + b = c,}$$

then we can derive other relationships between them. For instance, add $-b$ to both sides of this equation. We get

$$a + b - b = c - b,$$

whence $a + 0 = c - b$, or in other words,

$$\boxed{a = c - b.}$$

Similarly, we conclude that

$$\boxed{b = c - a.}$$

For instance, if

$$x + 3 = 5,$$

then

$$x = 5 - 3 = 2.$$

If

$$4 - a = 3,$$

then adding a to both sides yields

$$4 = 3 + a,$$

and subtracting 3 from both sides yields

$$1 = a.$$

If

$$-2 - y = 5,$$

then

$$-7 = y \quad \text{or} \quad y = -7.$$

EXERCISES

Justify each step, using commutativity and associativity in proving the following identities.

1. $(a + b) + (c + d) = (a + d) + (b + c)$
2. $(a + b) + (c + d) = (a + c) + (b + d)$
3. $(a - b) + (c - d) = (a + c) + (-b - d)$
4. $(a - b) + (c - d) = (a + c) - (b + d)$
5. $(a - b) + (c - d) = (a - d) + (c - b)$
6. $(a - b) + (c - d) = -(b + d) + (a + c)$
7. $(a - b) + (c - d) = -(b + d) - (-a - c)$
8. $((x + y) + z) + w = (x + z) + (y + w)$
9. $(x - y) - (z - w) = (x + w) - y - z$
10. $(x - y) - (z - w) = (x - z) + (w - y)$
11. Show that $-(a + b + c) = -a + (-b) + (-c)$.
12. Show that $-(a - b - c) = -a + b + c$.
13. Show that $-(a - b) = b - a$.

Solve for x in the following equations.

14. $-2 + x = 4$
15. $2 - x = 5$
16. $x - 3 = 7$
17. $-x + 4 = -1$
18. $4 - x = 8$
19. $-5 - x = -2$
20. $-7 + x = -10$
21. $-3 + x = 4$

22. Prove the **cancellation law for addition:**

$$\boxed{\text{If } a + b = a + c, \text{ then } b = c.}$$

23. Prove: If $a + b = a$, then $b = 0$.

§3. RULES FOR MULTIPLICATION

We can multiply integers, and the product of two integers is again an integer. We shall list the rules which apply to multiplication and to its relations with addition.

We again have the rules of ***commutativity*** and ***associativity:***

$$\boxed{ab = ba} \quad \text{and} \quad \boxed{(ab)c = a(bc).}$$

We emphasize that these apply whether a, b, c are negative, positive, or zero. Multiplication is also denoted by a dot. For instance

$$3 \cdot 7 = 21,$$

and

$$\begin{aligned} (3 \cdot 7) \cdot 4 &= 21 \cdot 4 = 84, \\ 3 \cdot (7 \cdot 4) &= 3 \cdot 28 = 84. \end{aligned}$$

For any integer a, the rules of multiplication by 1 and 0 are:

N6. $$\boxed{1a = a} \quad \textit{and} \quad \boxed{0a = 0.}$$

Example. We have

$$\begin{aligned} (2a)(3b) &= 2(a(3b)) \\ &= 2(3a)b \\ &= (2 \cdot 3)ab \\ &= 6ab. \end{aligned}$$